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Diffusion Equation Generalized for Modeling of Chladni Patterns

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Random walk of particles during Chladni pattern formation is macroscopically treated as a diffusion process. The corresponding generalized diffusion equation is formulated based upon the generator of vibration driven random walk by following Einstein's treatment of Brownian motion.

Keywords: Chladni patterns, vibration driven random walk, diffusion process

Highlights

- *Formation of Chladni patterns is macroscopically treated as a diffusion process.*
- *The diffusion coefficient is expressed by the parameters of driven random walk generator.*
- *The corresponding generalized diffusion equation is formulated.*
- *A generalization of Einstein's description of random walk phenomenon is performed.*
- *A new basis for the macroscopic description of Chladni pattern formation is given.*
- *A very complex physical phenomenon is rather simply described.*

1 INTRODUCTION

Experimental research of sound and mechanical vibrations by Ernst Chladni at the end of the eighteenth century was supported by observation of patterns formed by tiny particles on vibrating surfaces [1] and [2]. Although this phenomenon has significantly contributed to the development of acoustics, its analytical model has been formulated only recently [3]. For this purpose trajectories of bouncing particles were optically recorded, numerically analyzed and described by a new model of vibration driven random walk [3] and [4]. In this walk the horizontal displacement $\Delta\mathbf{r}=(\Delta x, \Delta y)$ during a single bounce is comprised of two statistically independent and normally distributed stochastic components. Since the standard deviation of each component is approximately proportional to the amplitude $A(\mathbf{r})$ of the surface vibration at the position $\mathbf{r}=(x,y)$, the generator of the corresponding random walk displacement $\Delta\mathbf{r}$ can be modeled by the Eq. (1):

$$\Delta\mathbf{r} = C A(\mathbf{r})\,\mathbf{G}. \tag{1}$$

Here $\mathbf{G}=(G(s_x), G(s_y))$ denotes the Gaussian random vector generator comprised of two Gaussian random number generators with seeds (s_x, s_y). The constant C determines the relation between the vibration amplitude A and the magnitude of the horizontal displacement $\Delta\mathbf{r}$ that can be statistically characterized by its probability distribution [3] and [4].

Applicability of the random walk model in Eq. (1) is here demonstrated by calculating the pattern formed on a square plate that is initially uniformly covered by the sand particles and then excited in the middle of its edge. Fig. 1 shows the distribution of the vibration amplitude $A(\mathbf{r})$ and some typical trajectories of particles during formation of the Chladni pattern, while Fig. 2 shows the evolution of pattern distribution. Its properties are in a good agreement with examples published in the literature about Chladni patterns and demonstrations of vibrations excited by sound [1] to [5].

Development of particle distribution in Fig. 2 reveals that during the formation of a Chladni pattern the particles are on average moving from intensively vibrating regions of the plate to the calm regions at the nodal lines. This happens because the random walk displacement $\Delta\mathbf{r}$ is on average proportional to the vibration amplitude $A(\mathbf{r})$ as described by Eq. (1). Consequently, the initially uniform distribution of particles is transformed into a pattern that exhibits the nodal lines. Similarly as in this case also several other Chladni patterns simulated by Eq. (1) reveal remarkably good agreement with corresponding experimental observations [3] to [5].

2 MACROSCOPIC DESCRIPTION OF CHLADNI PATTERN FORMATION

Since the random walk of bouncing particles resembles the Brownian motion, it can be macroscopically described by the diffusion equation as formulated by Einstein [6]. Transition from the microscopic description of a single particle movement to the

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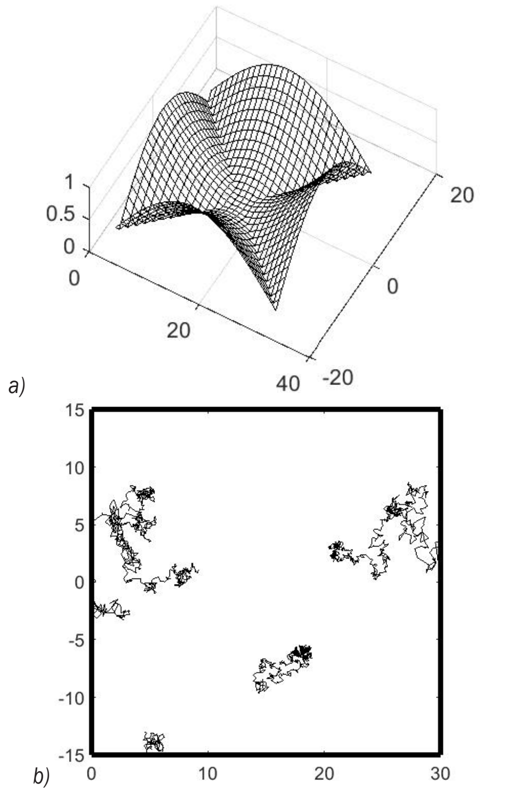

Fig. 1. *a) Distribution of the vibration amplitude $A(\mathbf{r})$, and b) typical trajectories comprised of 1000 random walk jumps of particles during the formation of the Chladni pattern (shown in Fig. 2)*

macroscopic description of the particle distribution is based upon the relation between the mean square displacement of particles $<|\Delta\mathbf{r}|^2>$ in the time interval τ and the diffusion coefficient $D=<|\Delta\mathbf{r}|^2>/2\tau$. Our model in Eq. 1 characterizes a single displacement during the vibration period τ by the mean square value $<|\Delta\mathbf{r}|^2>$ proportional to $A(\mathbf{r})^2$.

Consequently, the diffusion coefficient:

$$D(\mathbf{r})=C^2 A(\boldsymbol{r})^2<|\boldsymbol{G}|^2>/2\tau, \qquad (2)$$

depends on the position $\mathbf{r}$ and the evolution of the particle density $\rho(\mathbf{r},t)$ in time t has to be described by the diffusion equation of the generalized form:

$$\partial\rho(\mathbf{r},t)/\partial t=\operatorname{div}[D(\boldsymbol{r})\operatorname{grad}\rho(\boldsymbol{r},t)]. \qquad (3)$$

This equation is applicable for the determination of the particle density $\rho(\mathbf{r},t)$ during formation of a Chladni pattern when the geometric form of the plate and the amplitude distribution $A(\mathbf{r})$ are given. For this purpose a numerical treatment is generally needed since the diffusion coefficient $D(\boldsymbol{r})$ is not a constant.

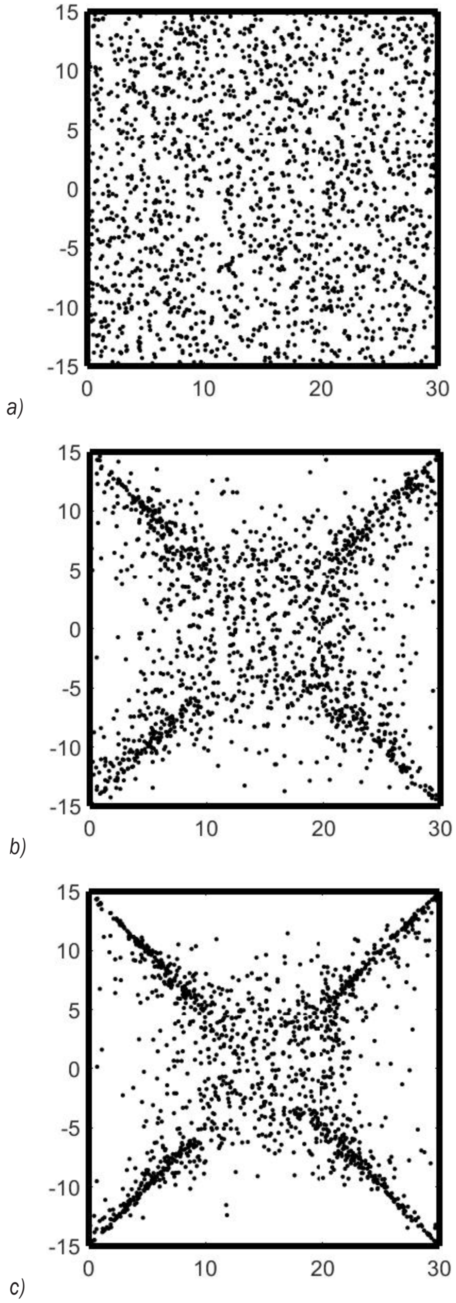

Fig. 2. *Chladni pattern formed from 2000 particles by a) 0 jumps, b) 500 jumps and c) 1000 jumps*

3 DISCUSION AND CONCLUSIONS

Some properties of very complex diffusion process during the formation of Chladni patterns can be predicted based upon general properties of the vibration driven random walk. Since the diffusion coefficient $D(\mathbf{r})$ is proportional to the square of the vibration amplitude $A(\mathbf{r})^2$ the particles diffuse from the regions of high amplitude to the regions of low amplitude. Consequently, the initially uniform distribution of particles is changed as indicated by the evolving Chladni pattern. At nodal lines the vibration amplitude $A(\mathbf{r})$ as well as the diffusion coefficient $D(\mathbf{r})$ is negligible, and therefore the diffusing particles are there accumulating. From the knowledge of nodal lines some basic characteristics of the particle density $\rho(\mathbf{r},t)$ in an evolving Chladni pattern can thus be predicted without solving the corresponding generalized diffusion equation.

When the surface vibration is caused by the interference of travelling waves the Chladni pattern can also be utilized to describe the properties of wave interference [7]. However, this is a bit surprising, since such a treatment corresponds to the macroscopic description of wave interference by the generalized diffusion equation. We therefore expect that a more detailed description of the phenomenon is needed for the modeling of random walk driven by the travelling waves. For this purpose a consideration of quantum-mechanical description of particle movement could be helpful.

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